Unraveling Euler's Number: Historical Perspectives and Modern Calculation Methods

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Abstract

In this paper, we explore multiple methods of calculating Euler's number e, including its limit definition, series expansion, continued fraction representation, and numerical methods. We analyze how each approach yields different insights and compare their efficiency in approximating e with help from matlab.

Contents

1 Introduction

This paper's aim is to explore the other concepts that aren't as common, from its origins to present. Euler's number has many uses and is seen in different aspects of mathematics. Many things would not be possible if it weren't for the existence of Euler's number. In this paper we will explore why that is the case.

- The Limit Definition: $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$
- Series Expansion: Using the Maclaurin series for e^x
- Continued Fraction Representation: A continued fraction form of e
- Numerical Methods: Techniques such as Newton's method and the Power Ratio Method

In the following sections, we will explore each of these concepts and their corresponding methods in detail.

2 The Limit Definition of e

Euler's number was first introduced by Jacob Bernoulli in 1683 when studying compound interest. It can be formally defined by Leonhard Euler in 1748 as the limit:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828459045\ldots \tag{1}$$

This definition connects e to the concept of continuous growth. The limit describes the behavior of compound interest as the number of compounding periods approaches infinity.

2.1 Approach 1: Limit Definition of e

Approach 1 involves calculating e using its limit definition. We begin by considering the expression $\left(1 + \frac{1}{n}\right)^n$ and evaluating it for increasing values of n.

2.1.1 Example: Limit Definition Approximation of e

- Step 1: Start with $n = 10$, calculate $\left(1 + \frac{1}{10}\right)^{10} = 2.59374$
- Step 2: Increase n to 100, calculate $\left(1 + \frac{1}{100}\right)^{100} = 2.70481$
- Step 3: For larger values of n, say $n = 1000$, the result approaches 2.71692
- Step 4: As n continues to grow, the expression converges to 2.71828

2.1.2 Example: Calculating e in Matlab Using the Limit

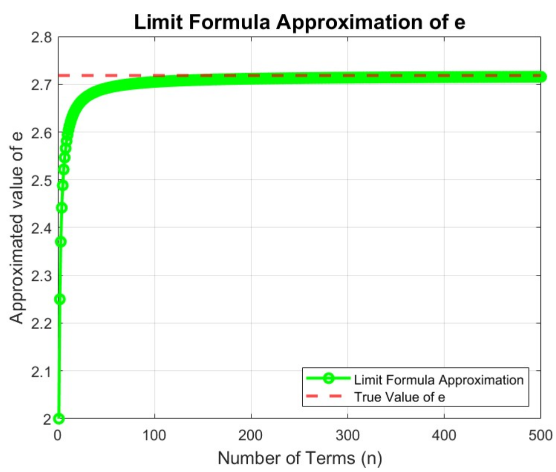

Figure 1: The Limit formula is calculated until n = 500 graphed against the true value of e.

```
490     2.715513250403941
491     2.715518878549868
492     2.715524483859836
493     2.715530066472583
494     2.715535626525330
495     2.715541164154643
496     2.715546679495975
497     2.715552172683664
498     2.715557643851103
499     2.715563093129869
500     2.715568520651728

Final approximation of e using Limit Formula with N=500: 2.715568520651728
```

Figure 2: This table shows us that it takes many terms to get close to the true value of e.

2.1.3 Example: Analyzing the Error of the Limit Definition of e in Matlab

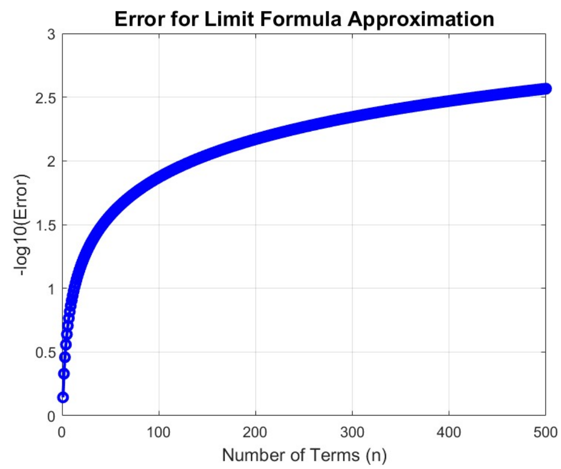

Figure 3: The graph shows us the error for the Limit Definition. This method has an accuracy to about 2.5 decimal places.

3 Series Expansion of e

Another method for calculating e is by using the **Maclaurin series** for e^x. This was developed by Issac Newton in 1669. This is given by:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2}$$

For $x = 1$, we obtain the series expansion for e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

3.1 Approach 2: Series Expansion for e

This approach calculates e by summing the terms of the series. The more terms we include, the closer the approximation becomes to the actual value of e.

3.1.1 Example: Series Approximation of e

- Step 1: Start with the first few terms of the series:

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2.70833$$

- Step 2: Add more terms for greater accuracy:

$$e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.70833$$

3.1.2 Example: Calculating e in Matlab Using The Maclaurin Series

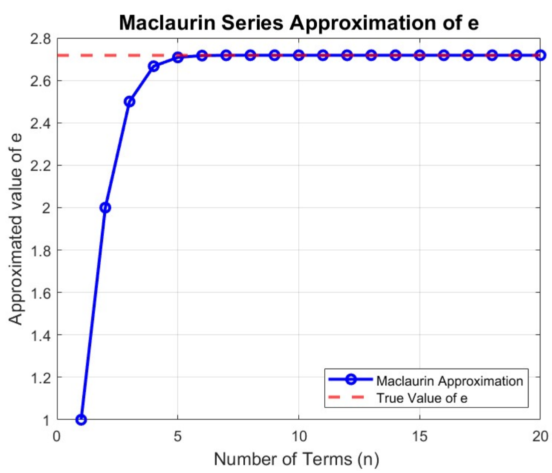

Figure 4: This graph shows us the Maclaurin Series approximation compared to the true value of e.

```
Maclaurin Series approximation for n=15: 2.718281828458230
Maclaurin Series approximation for n=16: 2.718281828458995
Maclaurin Series approximation for n=17: 2.718281828459043
Maclaurin Series approximation for n=18: 2.718281828459046
Maclaurin Series approximation for n=19: 2.718281828459046
Maclaurin Series approximation for n=20: 2.718281828459046
Final approximation of e using Maclaurin series with N=20: 2.718281828459046
```

Figure 5: In Matlab's command window we show that it takes less terms to get close to e from observing the graph at approximaitely n = 20.

3.1.3 Example: Analyzing the Error of the The Maclaurin Series of e in Matlab

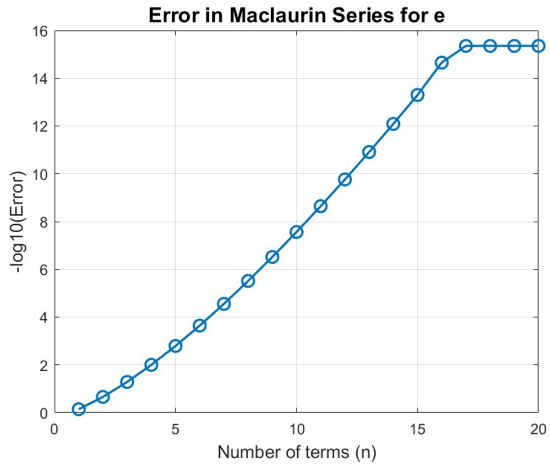

Figure 6: The graph shows us the error of Macluarin Series approximation of e. This method is accurate to about 15.3 decimal places.

4 Continued Fraction Representation of e

Before Euler formally defined e through the limit, he worked on representing e as a continued fraction in 1744. The continued fraction for e is given by:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}} \tag{3}$$

4.1 Approach 3: Continued Fractions for e

We can approximate e by truncating the continued fraction after a few terms.

4.1.1 Example: Continued Fractions Representation of e

- Step 1: Start with the first term:

$$e \approx 2 + \frac{1}{1} = 3$$

- Step 2: Add the second term:

$$e \approx 2 + \frac{1}{1+\frac{1}{2}} = 2.6667$$

- Step 3: Add the third term:

$$e \approx 2 + \frac{1}{1+\frac{1}{2+\frac{1}{1}}} = 2.718$$

By continuing the process, we can obtain increasingly accurate approximations of e. After several more terms, the value of e will eventually converge to 2.71828.....

4.1.2 Example: Calculating e in Matlab Using the Continued Fractions Method

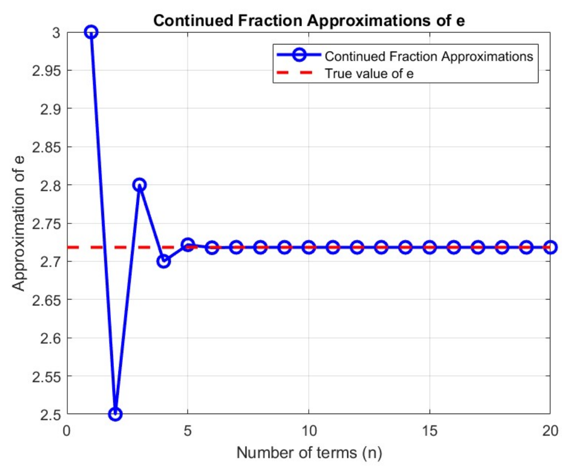

Figure 7: The graph shows the convergence of the Continued Fractions approximation. It seems different than the previous methods as the graph seems to have alternating signs right before it converges.

4.1.3 Example: Analyzing the Error of the Continued Fractions method of e in Matlab

Figure 8: The graph shows us the error for the Continued Fractions method. This method is accurate to about 15.3 decimal places.

5 Numerical Methods for Approximating e

The application of Newton's method in numerical methods is a modern technique in approximation. While it may not be efficient, it conveniently can approximate Euler's number e for us. We achieve this by solving the equation $f(x) = ln(x) - 1 = 0$, starting with an initial guess x_0.

5.1 Approach 4: Newton's Method to Calculate for e

We can use Newton's method to iteratively solve for e.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

5.1.1 Example: Newton's Method of Approximation for e

By using a numerical method like Newton's method, we can approximate e to several decimal places efficiently.

- Step 1: Choose an initial guess $x_0 = 2$
- Step 2: Apply the Newton method formula
- Step 3: Iterate the process to find successive approximations of e

$$x_0 = 2.0000, \quad x_1 = 2.6137, \quad x_2 = 2.7162$$

5.1.2 Example: Calculating e in Matlab using Newton's Method

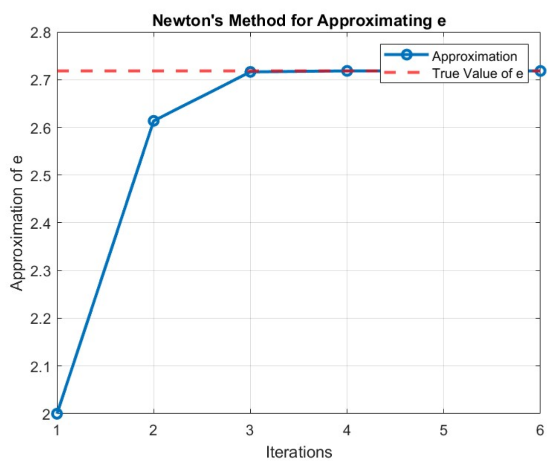

Figure 9: In this graph we show that there are a low amount of iterations performed to approximate e. It does not curve like the previous graphs. This is due to it converging quickly to e.

Iteration	Approximation of e
1	2.000000000000000
2	2.613705638880109
3	2.716243926355791
4	2.718281064358139
5	2.718281828458938
6	2.718281828459045

Figure 10: In this table we show how quickly newton's method converges. These are the same values used to graph Figure 9.

5.1.3 Example: Analyzing the Error of Newton's Method of Approximation for e in Matlab

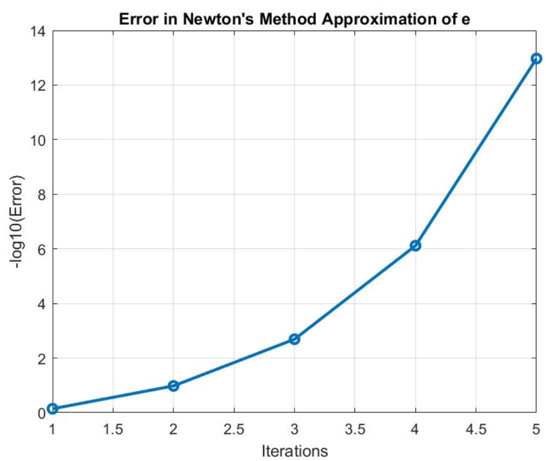

Figure 11: The graph shows us the method is accurate to about 12.9 decimal places. This due to tolerance given in Matlab, basically telling the program to stop iterations once its within a range of the value of e.

6 More Numerical Methods

6.1 Approach 5: Power Ratio Method (PRM)

I discovered the Power Ratio Method in the Brothers and Knox article, *New closed-form approximations to the logarithmic constant e.* This method was discovered by investigating the behavior of numbers raised to their own power.When we examine the rate of change of the ratio between adjacent integer values of x that have been raised to the x power lead to the approximation of e.[2] This method is described as:

$$e \approx \frac{(x+1)^{x+1}}{x^x} - \frac{x^x}{(x-1)^{x-1}} \tag{5}$$

6.2 Example: Approximating e in Matlab Using The Power Ratio Method

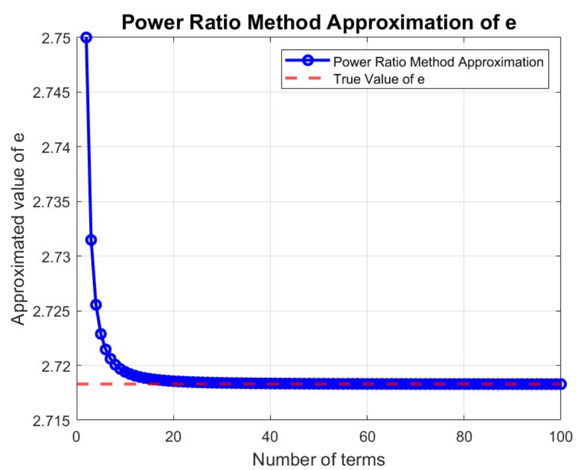

Figure 12: The Power Ratio Method converges relatively fast to e, but the difference with this method shows that it decays.

```
Power Ratio Method for x=96: 2.718294118694871
Power Ratio Method for x=97: 2.718293866582997
Power Ratio Method for x=98: 2.718293622150043
Power Ratio Method for x=99: 2.718293385086895
Power Ratio Method for x=100: 2.718293155100582
Final approximation of e using Power Ratio Method with x=100: 2.718293155100582
```

Figure 13: For this program of the Power Ratio Method we chose to go up to 100 terms, but early on at about term 18 the graph begins to converge to e.

6.3 Example: Analyzing the Error of Power Ratio Method Approximation for e in Matlab

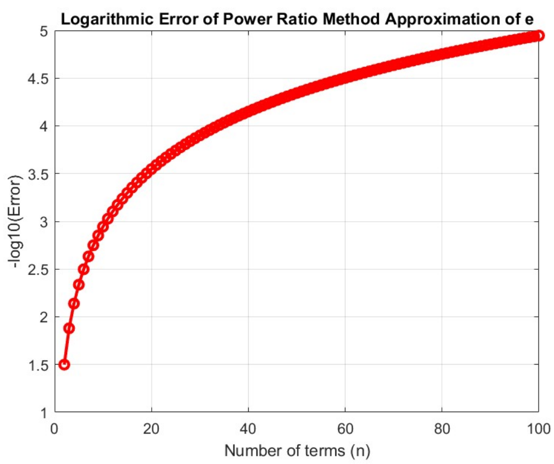

Figure 14: The graph of error shows us this method has an accuracy of up to 5 decimal places.

7 Comparison of Approaches

7.1 Methods

- Limit Definition: $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828459045\ldots$
- Series Expansion: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, when $x = 1$
- Continued Fraction: $e = 2 + \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1+\cdots}}}}$

- Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Power Ratio Method: $e \approx \frac{(x+1)^{x+1}}{x^x} - \frac{x^x}{(x-1)^{x-1}}$

7.2 Comparison of Methods in Matlab

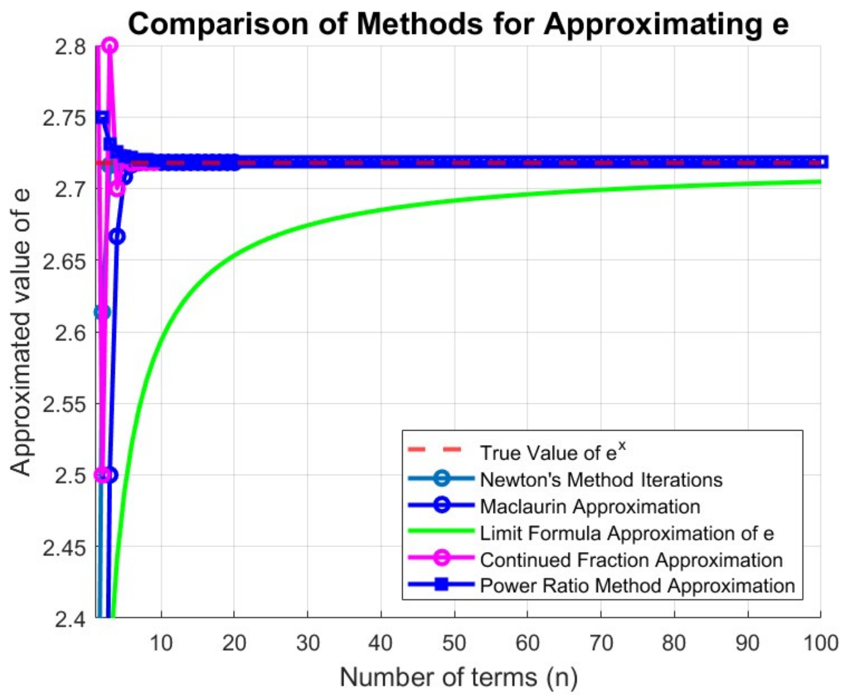

Figure 15: This graph shows us visually that certain methods are better in approximating e, because they use less terms to converging to e.

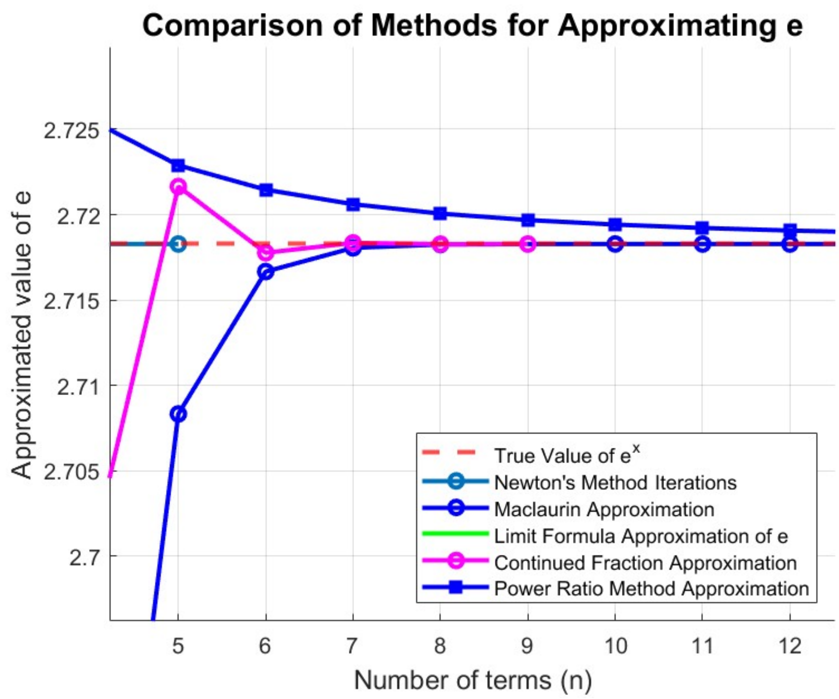

Figure 16: This graph shows us a closer look at the convergence of the methods.

8 Conclusion

In this research, we explored multiple methods to approximate Euler's number e, by analyzing each method's convergence characteristics and computational efficiency. We investigated approaches that include the limit definition of e, Maclaurin series expansion of e, the continued fraction representation of e, Newton's method for approximating e, and the power ratio method. By implementing these methods in Matlab, we were able to examine the progression of each approximation term by term, alongside the adjacent values for e at each step. Our graphical analysis helps visualize the different convergence rates. Some methods, such as the limit definition of e, offers us straightforward calculations but converge more slowly than others. In contrast, Newton's method, Power Ratio method and the Maclaurin series demonstrated faster convergence when

compared to the limit definition of e. The power ratio method introduced an alternative perspective by incorporating more complicated exponential behaviors in sequences, providing a unique approach to estimating e, and that it was a decay. Through this comparative analysis, we found that methods involving iterative or more complicated terms in the equations gave us better convergence on approximating e.

9 References

References

[1] McCartin, B.J. ”e: The Master of All.” The Mathematical Intelligencer 28, no. 2 (2006): 10–21. https://doi.org/10.1007/BF02987150.

[2] Brothers, Harlan J, and John A Knox. “New Closed-Form Approximations to the Logarithmic Constant e.” The Mathematical Intelligencer 20, no. 4 (1998): 25–29. https://doi.org/10.1007/BF03025225.